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The equation $X\nabla \det X = \det X \nabla \mathrm{tr}\, X$, multiplication of cofactor pair systems, and the Levi-Civita equivalence problem

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Abstract

Cofactor pair systems generalize the separable potential Hamiltonian systems. They admit n quadratic integrals of motion, they have a bi-Hamilton formulation, they are completely integrable and they are equivalent to separable Lagrangian systems. Cofactor pair systems can be constructed through a peculiar multiplicative structure of the so-called quasi-Cauchy–Riemann equations $(\mathrm{cof}\, J)^{-1}\nabla V = (\mathrm{cof}\, \tilde{J})^{-1}\nabla \tilde{V}$, where J and $\tilde{J}$ are special conformal Killing tensors.

In this work we have isolated the properties that are responsible for the multiplication, allowing us to give an elegant characterization of systems that admit multiplication. In this characterization the equation $X\nabla \det X = \det X \nabla \mathrm{tr}\, X$ plays a central role.

We describe how multiplication of quasi-Cauchy–Riemann equations can be considered as a special case of a more general kind of multiplication, defined on the solution space of certain systems of partial differential equations. We investigate algebraic properties of this multiplication and provide several methods for constructing new systems with multiplicative structure. We also discuss the role of the multiplication in the theory of equivalent dynamical systems on Riemannian manifolds, developed by Levi-Civita.

Keywords: Killing tensor; Nijenhuis torsion; Cauchy–Riemann equations; Separation of variables

1. Introduction

Throughout this paper, we will assume that we are on a n-dimensional Riemannian manifold Q with a metric (g_{ij}) and local coordinates (q^i). We study equations of the form

$$A^{-1}\nabla V = \tilde{A}^{-1}\nabla \tilde{V}, \tag{1}$$

where A and $\tilde{A}$ are non-singular tensors of rank two and ∇ is the gradient operator defined by $(\nabla V)^i = g^{ij}\partial_j V$ ($\partial_j := \partial/\partial q^j$). This is a system of n first-order linear partial differential equations in two dependent variables V, $\tilde{V}$ and n independent variables q^i. Whenever $n > 2$, the number of unknown functions is less than the number of

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equations and the system (1) is overdetermined. Thus, we cannot expect that non-trivial (non-constant) solutions exist without some restrictions on A and $\tilde{A}$.

Our main reason for studying systems of this kind is their connection to the so-called cofactor pair systems, which are certain kinds of dynamical systems on Riemannian manifolds.

By a dynamical system we mean a system of first-order equations equivalent to a Newton equation $\ddot{q}^h + \Gamma^h_{ij}\dot{q}^i\dot{q}^j = F^h$ with an external force $F = (F^h)$ that has no explicit dependence on time or velocity, and Γ^h_{ij} is the Christoffel symbol. Such a system is characterized by the triple (Q, g, F). A cofactor pair system is a dynamical system where the force F has two essentially different cofactor formulations, i.e., $F = -(\operatorname{cof} J)^{-1}\nabla V = -(\operatorname{cof} \tilde{J})^{-1}\nabla \tilde{V}$, where J and $\tilde{J}$ are two linearly independent non-singular special conformal Killing (SCK) tensors.

The theory of cofactor pair systems was developed in Linköping by Rauch-Wojciechowski, Marciniak, Lundmark, and Waksjö in [10,8,9,11]. In these papers, which deal with the Euclidean case, it has been proven that a cofactor pair system admits n quadratic integrals of motion, it has a bi-Hamiltonian formulation and it is completely integrable. These results were later generalized to a general Riemannian manifold in [3]. In [1] Benenti shows that cofactor pair systems are equivalent to separable Lagrangian systems in the sense of Levi-Civita. All these properties make cofactor pair systems a very interesting class of dynamical systems. For fixed SCK tensors J and $\tilde{J}$, a cofactor pair system is characterized by solutions of Eq. (1) with $A = \operatorname{cof} J$ and $\tilde{A} = \operatorname{cof} \tilde{J}$. When A and $\tilde{A}$ are the cofactors of SCK tensors, we call the corresponding equation

$$(\operatorname{cof} J)^{-1}\nabla V = (\operatorname{cof} \tilde{J})^{-1}\nabla \tilde{V} \tag{2}$$

a quasi-Cauchy–Riemann (QCR) equation.

The QCR equation may have constant or q dependent tensors A and $\tilde{A}$. In the case when A and $\tilde{A}$ are constant, on performing a suitable linear transformation $q \to Bq$, Eq. (1) takes the form $\nabla V = M\nabla\tilde{V}$, where M is in real Jordan form. An explicit formula for the general solution of the system $\nabla V = M\nabla\tilde{V}$ is presented by Jodeit and Olver in [5]. When $n = 2$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3}$$

the system (1) becomes the well known Cauchy–Riemann (CR) equations. For the CR equations, the ordinary multiplication of analytic functions can be interpreted as a bi-linear operation on the linear space of solutions. If $f = V + \mathrm{i}\tilde{V}$ and $g = W + \mathrm{i}\tilde{W}$ are two analytic functions, then the product $fg = (VW - \tilde{V}\tilde{W}) + \mathrm{i}(V\tilde{W} + \tilde{V}W)$ is also analytic. So for solutions of the CR equations, we can introduce a multiplication operation $*$ in the following way:

$$(V, \tilde{V}) * (W, \tilde{W}) = \left(VW - \tilde{V}\tilde{W}, V\tilde{W} + \tilde{V}W\right). \tag{4}$$

In other words, given two solutions $(V, \tilde{V})$ and $(W, \tilde{W})$ of the Cauchy–Riemann equations, the operation $*$, defined by (4), prescribes a new solution in a bi-linear way.

Lundmark discovered a very remarkable multiplicative structure on the class of cofactor pair systems [7]. Expressed as an operation $*$ on the solution space of Eq. (2) (in the case $n = 2$) this multiplication takes the form

$$(V, \tilde{V}) * (W, \tilde{W}) = \left(VW - (\det X)\tilde{V}\tilde{W}, V\tilde{W} + \tilde{V}W - (\operatorname{tr} X)\tilde{V}\tilde{W}\right),$$

where $X = \tilde{J}^{-1}J$. In the Euclidean case, when we choose $J = A$ and $\tilde{J} = \tilde{A}$ in (3), Eq. (2) reduces to the ordinary Cauchy–Riemann equations and $\det X = 1$, $\operatorname{tr} X = 0$. Thus, multiplication of analytic functions is reconstructed by this more general multiplication, which is well defined for cofactor pair systems. This is why we call the equation $(\operatorname{cof} J)^{-1}\nabla V = (\operatorname{cof} \tilde{J})^{-1}\nabla\tilde{V}$ a quasi-Cauchy–Riemann equation. A similar but more complicated multiplication formula exists for $n > 2$.

The existence of a multiplication is a very unusual and strong property of the QCR equation. Given a pair of solutions $(V, \tilde{V})$ and $(W, \tilde{W})$, the product $(V, \tilde{V}) * (W, \tilde{W})$ defines a new solution in a purely algebraic way. Hence, by forming polynomials and even convergent power series $\sum_k (V, \tilde{V})^k$ with respect to $*$, it is possible to generate large families of solutions (and thereby large families of cofactor pair systems) from one known solution $(V, \tilde{V})$.

Our aim is to investigate the multiplication $*$ in order to gain better understanding of its properties and of which systems allow this kind of multiplication. It turns out that the equation $X\nabla \det X = \det X \nabla \operatorname{tr} X$, appearing in the title of this work, plays an important role in characterizing systems of first-order PDE's that allow multiplication. Therefore, a significant part of this paper will be devoted to the study of this equation.

Remark 1. We adopt the usual convention for raising and lowering indices with the metric, e.g., $A^{ij} = A^i_k g^{kj}$ (summation over repeated indices is understood as usual and $[g^{ij}] = [g_{ij}]^{-1}$), and consider A_{ij}, A^{ij} and A_i^j as different versions of the same rank two tensor A. A rank two tensor A acts on a vector $X = (X^i)$ and on a 1-form $\alpha = (\alpha_i)$ as linear operators according to

$$(AX)^i = A^i_j X^j \quad \text{and} \quad (A\alpha)_i = A_i^j \alpha_j,$$

respectively. If A and B are tensors of rank two, we let AB denote the product of A and B when considered as linear operators acting on vectors, i.e.,

$$(AB)^i_j = A^i_k B^k_j.$$

2. Quasi-Cauchy–Riemann equations

A primary model for equations with multiplication is the *Quasi-Cauchy–Riemann* (QCR) equation

$$(\operatorname{cof} J)^{-1}\nabla V = (\operatorname{cof}\tilde{J})^{-1}\nabla \tilde{V}, \tag{5}$$

where J and $\tilde{J}$ are special conformal Killing tensors and $\operatorname{cof} J := (\det J)J^{-1}$.

2.1. Special conformal Killing tensors

We recall that a symmetric rank two tensor J is a *Killing tensor* if $\nabla_{(h}J_{ij)} = 0$, a *conformal Killing tensor* if $\nabla_{(h}J_{ij)} = \alpha_{(h}g_{ij)}$ and a *special conformal Killing* (SCK) *tensor* if $\nabla_h J_{ij} = \frac{1}{2}(\alpha_i g_{jh} + \alpha_j g_{ih})$ for a 1-form α (∇_h denotes the covariant derivative with respect to the Levi-Civita connection). If we contract the indices i and j, we see that the only possible choice for the 1-form is $\alpha = \mathrm{d}(\operatorname{tr} J)$ for a SCK tensor. It follows immediately that every SCK tensor is a conformal Killing tensor of trace type. Moreover, $\operatorname{cof} J$ is a Killing tensor whenever J is a SCK tensor. The metric g is always a SCK tensor, thus, every Riemannian manifold admits at least a one-dimensional space of SCK tensors. Depending on the metric g, additional SCK tensors may or may not exist. In the Euclidean space, there exist many SCK tensors, since J is a SCK tensor if and only if it has the form

$$J^{ij} = aq^iq^j + b^iq^j + b^jq^i + c^{ij},$$

where $a, b^i, c^{ij} = c^{ji}$ are arbitrary constants and q are Cartesian coordinates. Thus, the space of SCK tensors has the dimension $(n+1)(n+2)/2$. On the other hand, if a Riemannian manifold admits two independent SCK tensors and at least one of them has functionally independent eigenfunctions, then the manifold must be of constant curvature (see [2]). Hence, in a manifold with non-constant curvature, SCK tensors are rare.

2.2. The QCR equation

A Newton equation $\ddot{q}^h + \Gamma^h_{ij}\dot{q}^i\dot{q}^j = F^h$ admits an integral of motion $E = \frac{1}{2}A_{ij}\dot{q}^i\dot{q}^j + V(q)$ (quadratic in velocity), where the tensor A is non-singular, if and only if $F = -A^{-1}\nabla V$ and A is a Killing tensor. Thus, since $\operatorname{cof} J$ is a Killing tensor if J is a SCK tensor, the QCR equation (5) is a condition for the Lagrange equation to admit two integrals of motion $E = \frac{1}{2}(\operatorname{cof} J)_{ij}\dot{q}^i\dot{q}^j + V(q)$ and $\tilde{E} = \frac{1}{2}(\operatorname{cof}\tilde{J})_{ij}\dot{q}^i\dot{q}^j + \tilde{V}(q)$, that are quadratic in velocity. However, we will forget about the connection to dynamical systems and study the QCR equation (5) for itself.

The QCR equation can also be written as

$$X\nabla V = \det X\nabla\tilde{V},$$

where $X = \tilde{J}^{-1}J$.

2.3. The fundamental equation

Solutions of QCR equations can be characterized by solutions of a related system of linear second-order PDE's in one dependent variable, known as the fundamental equation. The fundamental equation can be expressed in a convenient way, using a differential operator D_J, associated with a SCK tensor J. This operator was introduced by Crampin and Sarlet in [3], and is defined as

$$D_J\theta = \frac{\mathrm{d}_J((\det J)\theta)}{\det J},$$

where d_J is another differential operator defined by the following properties:

(1) It acts on functions according to $\mathrm{d}_J f = J\mathrm{d}f$.
(2) $\mathrm{d}_J(\alpha \wedge \beta) = (\mathrm{d}_J\alpha) \wedge \beta + (-1)^k\alpha \wedge (\mathrm{d}_J\beta)$, when α is a k-form.
(3) d_J anti-commutes with the exterior derivative d, i.e., $\mathrm{d}_J\mathrm{d} + \mathrm{d}\mathrm{d}_J = 0$.

d_J is a derivation of degree 1 of type d_*, in the sense of Frölicher–Nijenhuis theory, see [4]. D_J is not a derivation but it has the following properties:

Proposition 2. (1) $D_J^2 = 0$.
(2) *Let θ be a k-form. Then $D_J\theta = 0$ if and only if there exists (locally) a $(k-1)$-form ϕ such that $\theta = D_J\phi$.*
(3) *If $\tilde{J}$ is another SCK tensor (of the same metric g) and μ is a real number, then*

$$D_{J+\mu\tilde{J}} = D_J + \mu D_{\tilde{J}} \quad \text{and} \quad D_J D_{\tilde{J}} + D_{\tilde{J}} D_J = 0.$$

The proof follows from corresponding properties of the operator d_J, that can be found in [13].

Theorem 3 (*Fundamental Equation*). *If $(V, \tilde{V})$ satisfies the QCR equation (5), then both functions $V/\det J$ and $\tilde{V}/\det\tilde{J}$ satisfy the same equation*

$$D_J D_{\tilde{J}}\phi = 0. \tag{6}$$

Conversely, each solution ϕ of Eq. (6) generates two solutions $(V, \tilde{V})$ of the QCR system, one with $V = (\det J)\phi$ and one with $\tilde{V} = (\det\tilde{J})\phi$.

Theorem 3 gives a 1–1 correspondence between solutions of the QCR equation and of the related fundamental equation. For each solution ϕ of the fundamental equation, there exists a unique (up to an irrelevant additive constant) solution $(V, \tilde{V})$, with $V = (\det J)\phi$.

2.4. The μ dependent QCR equation

In this study it turns out that it is more natural to consider the parameter dependent system

$$(\mathrm{cof}(J + \mu\tilde{J}))^{-1}\nabla V_\mu = (\mathrm{cof}\tilde{J})^{-1}\nabla\tilde{V}, \tag{7}$$

where μ is a real parameter. The system (7) is obtained by replacing J with $J + \mu\tilde{J}$ in the QCR equation. Due to the linear structure of SCK tensors, $J + \mu\tilde{J}$ is also a SCK tensor, thus (7) is in fact a parameter dependent QCR equation. Eq. (7) can be written as

$$X_\mu\nabla V_\mu = \det X_\mu\nabla\tilde{V}, \tag{8}$$

where $X_\mu = X + \mu I$ and $X = \tilde{J}^{-1}J$. The parameter dependent system (7) has $D_{J+\mu\tilde{J}}D_{\tilde{J}}\phi = 0$ as the related fundamental equation. Hence, according to Proposition 2, the fundamental equation is in fact the same as for the ordinary QCR equation $(\mathrm{cof}\ J)^{-1}\nabla V = (\mathrm{cof}\ \tilde{J})^{-1}\nabla\tilde{V}$. Thus, there is also a 1–1 correspondence between the solutions of a QCR equation and the solutions of the related parameter dependent QCR equation. Explicitly, for each solution $(V, \tilde{V})$ of the QCR equation (5), there exists a unique (up to constants) solution V_μ of the μ dependent QCR equation (7) such that $V^{(0)} = V$ and $V^{(n-1)} = \tilde{V}$. Passing to the equivalent parameter dependent QCR system is a key ingredient in the proof of the multiplication theorem for cofactor pair systems, presented in [7].

The fact that Eqs. (5) and (7) share the same fundamental equation is closely connected to the “2 implies n” theorem, which states that a dynamical system that allows two quadratic (in velocity) integrals $E = (\operatorname{cof} J)_{ij}\dot{q}^i\dot{q}^j + V(q)$ and $\tilde{E} = (\operatorname{cof}\tilde{J})_{ij}\dot{q}^i\dot{q}^j + \tilde{V}(q)$ of motion of cofactor type must in fact have n quadratic integrals of motion, see [8].

Remark 4. A special case of multiplication for QCR equations, expressed as a recursion formula, was discovered before the general multiplication was found. The recursion formula was first discovered in the Euclidean case for $n = 2$ in [10]. Later it was generalized to the general Euclidean case in [8] and thereafter to a general Riemannian manifold in [7].

In this paper, we will take this multiplication to its natural environment: the solution space of certain systems of partial differential equations. By doing this, we have been able to characterize systems that allow a more general multiplication, which includes the multiplication of cofactor pair systems as a special case.

3. Multiplication

Let X be a non-singular rank two tensor and let $X_\mu = X + \mu I$, where μ is a real parameter. Inspired by multiplication of cofactor pair systems, we consider more general systems $X_\mu \nabla V_\mu = \det X_\mu \nabla \tilde{V}$. In this section we intend to investigate under which restrictions this equation admits a multiplication on its space of solutions, that is similar to the one present for cofactor pair systems. The components in this vector equation are polynomials in μ and by analyzing the degrees, we realize that the dependent variable V_μ is a polynomial of degree $n-1$. We let $V^{(i)}$ denote the coefficient in V_μ at degree i, i.e., $V_\mu = V^{(0)} + V^{(1)}\mu + \cdots + V^{(n-1)}\mu^{n-1}$. By studying the highest order terms in the equation, we find that $\tilde{V}$ and $V^{(n-1)}$ must be the same up to an inessential constant term. Thus, we consider equations of the form

$$X_\mu \nabla V_\mu = \det X_\mu \nabla V^{(n-1)}, \tag{9}$$

or the equivalent form $X_\mu \mathrm{d}V_\mu = \det X_\mu \mathrm{d}V^{(n-1)}$. Let $Z^{(i)}$ be defined as $\det X_\mu = Z^{(0)} + Z^{(1)}\mu + \cdots + Z^{(n)}\mu^n$, i.e., $Z^{(i)}$ is the elementary symmetric polynomial of degree $n-i$ in the eigenvalues of X. We note that $Z^{(0)} = \det X$, $Z^{(n-1)} = \operatorname{tr} X$, $Z^{(n)} = 1$. By identifying coefficients of different powers of μ in (9), we obtain the equations

$$X\nabla V^{(i)} + \nabla V^{(i-1)} = Z^{(i)}\nabla V^{(n-1)}, \quad i = 0, \ldots, n-1, \tag{10}$$

where $V^{(-1)} := 0$. We observe that this is a system of n^2 first-order linear PDE’s for n dependent variables $V^{(i)}$. Since X is non-singular, it is possible to remove one of the vector equations in (10) by using the Cayley–Hamilton theorem, leaving us with a system of $n(n-1)$ PDE’s. Since the elements of $X_\mu \nabla V_\mu$ are polynomials in μ, which are of degree at most the degree of $\det X_\mu$, we see that Eq. (9) is also equivalent to

$$X_\mu \nabla V_\mu \equiv 0 \pmod{\det X_\mu}, \tag{11}$$

meaning that each component in the vector $X_\mu \nabla V_\mu$, when considered as a polynomial in μ, is divisible by the polynomial $\det X_\mu$. Each of the three different ways, (9)–(11), of writing the same system of PDE’s has its own advantages. The form (11) is most suitable for discussing the forthcoming multiplication. In the following, whenever the equivalence symbol $\equiv$ is used, it should be understood as equivalence modulo the polynomial $\det X_\mu$.

3.1. The $*$-operator

We introduce a multiplication $*$ on the space of solutions of (11), by defining the product $V_\mu * W_\mu$, of two solutions V_μ, W_μ, as the residue of the polynomial $V_\mu W_\mu$ after division by $\det X_\mu$. Given two solutions V_μ and W_μ, there exist unique polynomials Q_μ and $R_\mu = V_\mu * W_\mu$ such that

$$V_\mu W_\mu = Q_\mu \det X_\mu + R_\mu,$$

where the degree of R_μ is less than n.

The function $V_\mu * W_\mu$ is not in general a solution of the system (11). We have to consider a special kind of system of the form (11), where we put some restrictions on the tensor X. We will now investigate which restrictions are necessary and sufficient in order for $V_\mu * W_\mu$ to be a solution. For solutions V_μ, W_μ of Eq. (11), there exists a polynomial Q_μ such that $V_\mu * W_\mu = V_\mu W_\mu - Q_\mu \det X_\mu$. Thus, we have that

$$\begin{aligned} X_\mu \nabla (V_\mu * W_\mu) &= W_\mu X_\mu \nabla V_\mu + V_\mu X_\mu \nabla W_\mu - (\det X_\mu) X_\mu \nabla Q_\mu - Q_\mu X_\mu \nabla \det X_\mu \\ &\equiv 0 + 0 - 0 - Q_\mu X_\mu \nabla \det X_\mu \\ &= -Q_\mu X_\mu \nabla \det X_\mu. \end{aligned} \tag{12}$$

If we require that $V_\mu * W_\mu$ should be a solution whenever V_μ and W_μ are solutions, a necessary and sufficient condition is that $X_\mu \nabla \det X_\mu \equiv 0$. The sufficiency is obvious. To see the necessity, we choose V_μ and W_μ as the trivial solutions μ and μ^{n-1} respectively. Then

$$\begin{aligned} V_\mu W_\mu &= \mu \mu^{n-1} \\ &= \mu^n \\ &\equiv \det X_\mu - (Z^{(0)} + Z^{(1)} \mu + \cdots + Z^{(n-1)} \mu^{n-1}). \end{aligned}$$

Thus, $V_\mu * W_\mu = -(Z^{(0)} + Z^{(1)}\mu + \cdots + Z^{(n-1)}\mu^{n-1})$ and, more important, $Q_\mu = -1$. The calculation (12) then implies that $V_\mu * W_\mu$ is a solution if and only if $X_\mu \nabla \det X_\mu \equiv 0$. Thereby, we have proven that systems of the form (11) that allow the $*$-multiplication are characterized solely by solutions of the equation $X_\mu \nabla \det X_\mu \equiv 0$, which will be investigated in the next section. We can formulate this result as

Theorem 5 (*Multiplication*). *Assume that the tensor X satisfies the equation $X_\mu \nabla \det X_\mu \equiv 0$ and that V_μ, W_μ are two solutions of the equation $X_\mu \nabla V_\mu \equiv 0$. Then $V_\mu * W_\mu$ is also a solution of $X_\mu \nabla V_\mu \equiv 0$.*

In order to clarify how this multiplication works, we consider the simple case $n = 2$ in detail. Let $V_\mu = V + \mu \tilde{V}$ and $W_\mu = W + \mu \tilde{W}$ be two solutions of Eq. (11), which is equivalent to the equation $X \nabla V = \det X \nabla \tilde{V}$ when $n = 2$. The ordinary product of the polynomials V_μ and W_μ (not yet a solution of (11)) becomes

$$V_\mu W_\mu = VW + (V\tilde{W} + \tilde{V}W)\mu + \tilde{V}\tilde{W}\mu^2.$$

Since $\det X_\mu = \mu^2 + \operatorname{tr} X \mu + \det X$, we can reduce μ^2 modulo the polynomial $\det X_\mu$. We obtain $\mu^2 \equiv -(\operatorname{tr} X)\mu - \det X$. $V_\mu * W_\mu$ is by definition the unique polynomial that is linear in μ and congruent to the product $V_\mu W_\mu$. Hence,

$$\begin{aligned} V_\mu * W_\mu &= VW + (V\tilde{W} + \tilde{V}W)\mu + \tilde{V}\tilde{W}(-(\operatorname{tr} X)\mu - \det X) \\ &= VW - (\det X)\tilde{V}\tilde{W} + \left(V\tilde{W} + \tilde{V}W - (\operatorname{tr} X)\tilde{V}\tilde{W}\right)\mu. \end{aligned}$$

When $n = 2$, the multiplication formula becomes clearer when formulated without the μ-notation. Let $(V, \tilde{V})$ and $(W, \tilde{W})$ be two solutions of the equation $X \nabla V = \det X \nabla \tilde{V}$. Then

$$(V, \tilde{V}) * (W, \tilde{W}) := \left(VW - (\det X)\tilde{V}\tilde{W},\ V\tilde{W} + \tilde{V}W - (\operatorname{tr} X)\tilde{V}\tilde{W}\right) \tag{13}$$

is also a solution of the same equation.

Remark 6. The multiplication can be introduced for more general systems

$$A_\mu \nabla V_\mu \equiv 0 \pmod{P_\mu}, \tag{14}$$

where A_μ is an arbitrary rank two tensor that is a polynomial in μ, P_μ is a fixed polynomial in μ of degree m, V_μ is a polynomial of one degree less than P_μ. If A_μ and P_μ are related through the equation $A_\mu \nabla P_\mu \equiv 0 \pmod{P_\mu}$, solutions V_μ and W_μ of (14) can be multiplied in a way similar to the case described in Theorem 5. The reason why we have presented the multiplication for the more special kind of system, is to make the connection to cofactor pair systems (described later in this work) more visible.

The multiplication constitutes a powerful tool for generating, in a purely algebraic way, new solutions of the system (11). For example, we can start with a trivial solution, i.e., a polynomial in μ of degree at most $n - 1$ with constant coefficients, and multiply it by itself repeatedly. This procedure will usually generate an infinite sequence of non-trivial solutions. The next theorem contains some algebraic properties of the multiplication.

Theorem 7. *The space of solutions of (9) together with the scalar multiplication, addition (defined in the obvious way) and multiplication $*$ is an algebra over $\mathbb{R}$, where the $*$-multiplication is associative and commutative. Moreover, a solution V_μ has a multiplicative inverse in the space of solutions if and only if V_μ and* $\det X_\mu$ *are relatively prime.*

Proof. The linear structure of solutions and the bi-linearity of the multiplication are obvious. Thus, the space of solutions has the described algebra structure. The associative and commutative properties of the multiplication follow immediately from properties of multiplication in general quotient rings of polynomials. Suppose that V_μ is a solution, then there exists a polynomial $\tilde{V}_\mu$, of degree less than n, such that $V_\mu \tilde{V}_\mu \equiv 1$ if and only if V_μ and $\det X_\mu$ are relatively prime. We need only show that if such polynomial exists, it must be a solution of $X_\mu \nabla V_\mu \equiv 0$. There exists a unique polynomial Q_μ such that $V_\mu \tilde{V}_\mu = Q_\mu \det X_\mu + 1$. Thus

$$\begin{aligned} 0 &\equiv X_\mu \nabla 1 \equiv X_\mu \nabla (Q_\mu \det X_\mu + 1) = X_\mu \nabla (V_\mu \tilde{V}_\mu) \\ &= \tilde{V}_\mu X_\mu \nabla V_\mu + V_\mu X_\mu \nabla \tilde{V}_\mu \equiv V_\mu X_\mu \nabla \tilde{V}_\mu. \end{aligned}$$

Hence, the proof is complete. □

Example 8. The trivial solution $V_\mu = \mu$ has a multiplicative inverse if and only if μ is not a factor of $\det X_\mu$, i.e., $\det X \neq 0$. In that case

$$\mu^{-1} = -\frac{1}{Z^{(0)}}(Z^{(1)} + Z^{(2)}\mu + \cdots + Z^{(n)}\mu^{n-1}).$$

4. The equation $X\nabla \det X = \det X \nabla \operatorname{tr} X$

As we have seen, the multiplication introduced for the system $X_\mu \nabla V_\mu \equiv 0 \pmod{\det X_\mu}$, relies on the assumption that the tensor X satisfies the equation

$$X_\mu \nabla \det X_\mu \equiv 0 \pmod{\det X_\mu}, \tag{15}$$

that is equivalent to

$$X_\mu \nabla \det X_\mu = \det X_\mu \nabla \operatorname{tr} X_\mu,$$

that in components reads

$$X\nabla Z^{(i)} + \nabla Z^{(i-1)} = Z^{(i)} \nabla Z^{(n-1)}, \quad i = 0, \ldots, n-1, \tag{16}$$

where $Z^{(-1)} := 0$. When considered as equations for the tensor X, we see that the equations above form a system of $n(n-1)$ non-linear first-order PDE's for n^2 unknown functions $X^i_{\ j}$. The aim of this section is to investigate properties of Eq. (15).

4.1. X torsionless

The *Nijenhuis torsion* of a $(1, 1)$-tensor J is a $(1, 2)$-tensor, N_J, defined through its action on vector fields X and Y as

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y].$$

A tensor with vanishing torsion is called *torsionless*. The components of torsion tensor are given by $(N_J)^k_{ij} = 2(J^h_{[i} \nabla_{|h|} J^k_{j]} - J^k_m \nabla_{[i} J^m_{j]})$. In [3], it is proven that any torsionless tensor satisfies the equation $X\nabla \det X = \det X \nabla \operatorname{tr} X$. However, one can go further and express $X\mathrm{d}(\det X) - \det X\mathrm{d}(\operatorname{tr} X)$ as a contraction of the torsion with the cofactor tensor of X:

Proposition 9. *Any tensor X of rank* 2 *satisfies the relation*

$$2\,(X\mathrm{d}(\det X) - \det X\mathrm{d}(\operatorname{tr} X))_i = (N_X)^k_{ij} C^j_k.$$

Proof. By using the relation $\partial_k \det X = \nabla_k X^i_j C^j_i$, where $C = \mathrm{cof}(X) = (\det X)X^{-1}$, we obtain

$$\begin{aligned}(N_X)^k_{ij}C^j_k &= 2\left(X^h_i\nabla_h X^k_j - X^h_j\nabla_h X^k_i - X^k_h\nabla_i X^h_j + X^k_h\nabla_j X^h_i\right)C^j_k\\ &= 2\left(X^h_i\partial_h \det X - \det X\delta^h_k\nabla_h X^k_i - \det X\delta^j_h\nabla_i X^h_j + \det X\delta^j_h\nabla_j X^h_i\right)\\ &= 2\left(X\mathrm{d}(\det X) - \det X\mathrm{d}(\mathrm{tr}\, X)\right)_i . \quad \square\end{aligned}$$

Since X and $X + \mu I$ have the same torsion, we immediately get the following result:

Corollary 10. *Any torsionless tensor X satisfies (15).*

Thus, every torsionless tensor of rank two, generates a system of PDE's with a multiplicative structure on its solution space.

4.2. $X = \tilde{J}^{-1}J$

Any SCK tensor is torsionless and therefore, it satisfies Eq. (15) according to the previous theorem. More generally, we have:

Theorem 11. *Suppose that J and $\tilde{J}$ are SCK tensors and let $X = \tilde{J}^{-1}J$ ($X^i_j = (\tilde{J}^{-1})^i_k J^k_j$). Then X satisfies Eq. (15).*

The proof relies on some properties of equivalent metrics that are proven in [1]. We say that two metrics g and $\bar{g}$, on the same configuration space Q, are equivalent if their geodesics locally coincide when considered as unparametrized curves. There is a close connection between equivalence of metrics and SCK tensors. If J is a SCK tensor of g, then $\bar{g}^{ij} = c(\det J)J^{ij}$ is equivalent to g for every non-zero constant c. Moreover, every metric equivalent to g can be expressed in this way. There is also a 1–1 correspondence between non-singular SCK tensors of equivalent metrics. Namely, if $\bar{g}^{ij} = (\det J)J^{ij}$ for a SCK tensor J of g and $\tilde{J}$ is another SCK tensor of g, then $\bar{J} = \tilde{J}J^{-1}$ is a SCK tensor of $\bar{g}$.

Proof. Let $\bar{J} = J\tilde{J}^{-1}$ and let $\bar{g}$ be the metric equivalent to g, defined by $\bar{g}^{ij} = (\det\tilde{J})\tilde{J}^{ij}$. Let $\bar{\nabla}$ denote the covariant derivative with respect to the Levi-Civita connection of the metric $\bar{g}$. Then, since $\bar{J}$ is a SCK tensor of $\bar{g}$, we have

$$\begin{aligned}0 &= \left(\bar{J}\bar{\nabla}\det\bar{J} - \det\bar{J}\bar{\nabla}\mathrm{tr}\,\bar{J}\right)^i\\ &= \bar{J}^i_h\bar{g}^{hj}\partial_j\det\bar{J} - (\det\bar{J})\bar{g}^{ij}\partial_j\mathrm{tr}\,\bar{J}\\ &= \det\tilde{J}\left(J^i_r(\tilde{J}^{-1})^r_h\tilde{J}^h_k g^{kj}\partial_j\det\bar{J} - (\det\bar{J})\tilde{J}^i_k g^{kj}\partial_j\mathrm{tr}\,\bar{J}\right)\\ &= \det\tilde{J}\left(J^i_k g^{kj}\partial_j\det\bar{J} - (\det\bar{J})\tilde{J}^i_k g^{kj}\partial_j\mathrm{tr}\,\bar{J}\right)\\ &= \det\tilde{J}\left(J\nabla\det\bar{J} - (\det\bar{J})\tilde{J}\nabla\mathrm{tr}\,\bar{J}\right)^i .\end{aligned}$$

Thus, $J\nabla\det\bar{J} = (\det\bar{J})\tilde{J}\nabla\mathrm{tr}\,\bar{J}$, which is equivalent to $X\nabla\det X = \det X\nabla\mathrm{tr}\,X$. Since $J + \mu\tilde{J}$ is a SCK tensor whenever J and $\tilde{J}$ are, we can replace J with $J + \mu\tilde{J}$, which completes the proof. $\square$

According to Theorem 11 any QCR equation $(\mathrm{cof}J)^{-1}\nabla V = (\mathrm{cof}\tilde{J})^{-1}\nabla\tilde{V}$ is equipped with the multiplication described in Theorem 5. When restricted to QCR equations, Theorem 5 reduces to Lundmark's multiplication of cofactor pair systems. However, the multiplication theorem in this work ensures that multiplication exists for a much larger class of equations than just QCR equations.

4.3. New solutions from known solutions

The following two theorems tell us how to produce new solutions of Eq. (15) from known solutions.

Theorem 12. *Suppose that X is non-singular. Then X solves Eq. (15) if and only if X^{-1} solves the same equation. In other words,*

$$X_\mu \nabla \det X_\mu = \det X_\mu \nabla \mathrm{tr}\, X_\mu$$

if and only if

$$(X^{-1})_\mu \nabla \det(X^{-1})_\mu = \det(X^{-1})_\mu \nabla \mathrm{tr}\,(X^{-1})_\mu. \tag{17}$$

Proof. Define $W^{(i)}$ through

$$\det(X^{-1})_\mu = W^{(0)} + W^{(1)}\mu + \cdots + W^{(n)}\mu^n, \quad W^{(-1)} := 0.$$

Then X^{-1} satisfies Eq. (17) if and only if

$$0 = X^{-1}\nabla W^{(i)} + \nabla W^{(i-1)} - W^{(i)}\nabla W^{(n-1)} \quad i = 0, \ldots, n-2.$$

Let λ_i denote the eigenvalues of X, then the eigenvalues of X^{-1} are $1/\lambda_i$. Thus,

$$\begin{aligned} W^{(i)} &= \sum_{1 \le r_1 < \cdots < r_{n-i} \le n} \frac{1}{\lambda_{r_1}} \cdots \frac{1}{\lambda_{r_{n-i}}} \\ &= \sum_{1 \le r_1 < \cdots < r_i \le n} \frac{\lambda_{r_1} \cdots \lambda_{r_i}}{\lambda_1 \cdots \lambda_n} \\ &= \frac{Z^{(n-i)}}{Z^{(0)}}. \end{aligned}$$

Therefore, if we let $Z^{(n+1)} := 0$, we have

$$\begin{aligned} &X^{-1}\nabla W^{(i)} + \nabla W^{(i-1)} - W^{(i)}\nabla W^{(n-1)} \\ &\quad = X^{-1}\left(\nabla \frac{Z^{(n-i)}}{Z^{(0)}} + X\nabla \frac{Z^{(n-i+1)}}{Z^{(0)}} - \frac{Z^{(n-i)}}{Z^{(0)}} X\nabla \frac{Z^{(1)}}{Z^{(0)}}\right) \\ &\quad = \frac{X^{-1}}{(Z^{(0)})^2}\Bigg(Z^{(0)}\nabla Z^{(n-i)} - Z^{(n-i)}\nabla Z^{(0)} + Z^{(0)} X\nabla Z^{(n-i+1)} \\ &\qquad - Z^{(n-i+1)} X\nabla Z^{(0)} - Z^{(n-i)} X\nabla Z^{(1)} + \frac{Z^{(n-i)}Z^{(1)}}{Z^{(0)}} X\nabla Z^{(0)}\Bigg) \\ &\quad = \frac{X^{-1}}{(Z^{(0)})^2}\Bigg(Z^{(n-i+1)}(Z^{(0)}\nabla Z^{(n-1)} - X\nabla Z^{(0)}) - Z^{(n-i)}\nabla Z^{(0)} \\ &\qquad - Z^{(n-i)} X\nabla Z^{(1)} + \frac{Z^{(n-i)}Z^{(1)}}{Z^{(0)}} X\nabla Z^{(0)}\Bigg) \\ &\quad = \frac{Z^{(1)}Z^{(n-i)}X^{-1}}{(Z^{(0)})^3}\left(-Z^{(0)}\nabla Z^{(n-1)} + X\nabla Z^{(0)}\right) \\ &\quad = 0, \end{aligned}$$

where we have used the relation (16) repeatedly. □

One can also construct solutions of (15) from known solutions of smaller dimension.

Theorem 13 (*Direct Sum*). *Let $r = (q^1, \ldots, q^m)$, $s = (q^{m+1}, \ldots, q^n)$, and*

$$(\nabla_r)^i = \sum_{j=1}^{m} g^{ij}\partial_j, \quad (\nabla_s)^i = \sum_{j=m+1}^{n} g^{ij}\partial_j.$$

Suppose that A and B are rank two tensors such that $A_\mu \nabla_r \det A_\mu = \det A_\mu \nabla_r \operatorname{tr} A_\mu$ and $B_\mu \nabla_s \det B_\mu = \det B_\mu \nabla_s \operatorname{tr} B_\mu$, where the elements in A depend only on the coordinates in r and the elements in B only on the coordinates in s. Then

$$X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

satisfies Eq. (15).

Proof.

$$\begin{aligned}
\left(X_\mu \mathrm{d}(\det X_\mu)\right)_a &= (X_\mu)^i_a \partial_i \left(\det A_\mu \det B_\mu\right) \\
&= (\det B_\mu)(X_\mu)^i_a \partial_i \left(\det A_\mu\right) + (\det A_\mu)(X_\mu)^i_a \partial_i \left(\det B_\mu\right) \\
&= \sum_{i=1}^{m} (\det B_\mu)(A_\mu)^i_a \partial_i \left(\det A_\mu\right) + \sum_{i=m+1}^{n} (\det A_\mu)(B_\mu)^i_a \partial_i \left(\det B_\mu\right) \\
&= \det B_\mu \det A_\mu \partial_a (\operatorname{tr} A_\mu + \operatorname{tr} B_\mu) \\
&= \left(\det X_\mu \mathrm{d}(\operatorname{tr} X_\mu)\right)_a . \quad \square
\end{aligned}$$

Since each torsionless tensor solves Eq. (15), it is natural to ask whether the converse is true. Is every solution, X, of Eq. (15) torsionless? We have presented a few methods for generating new solutions of Eq. (15). As an attempt to answer the question above, we can check whether the generated solutions are torsionless. It is easy to see that a tensor X constructed by the method in Theorem 13 is torsionless if and only if the tensors A and B are torsionless. Moreover, if X is torsionless, by definition, we have

$$[Xu, Xv] - X[Xu, v] - X[u, Xv] + X^2[u, v] = 0 \tag{18}$$

for all vectors u and v. Let $\tilde{u} = Xu$ and $\tilde{v} = Xv$ and multiply Eq. (18) with X^{-2}. Then we obtain

$$X^{-2}[\tilde{u}, \tilde{v}] - X^{-1}[\tilde{u}, X^{-1}\tilde{v}] - X^{-1}[X^{-1}\tilde{u}, \tilde{v}] + [X^{-1}\tilde{u}, X^{-1}\tilde{v}] = 0.$$

Since $\tilde{u}$ and $\tilde{v}$ are arbitrary vectors, X^{-1} must be torsionless whenever X is torsionless. Hence, Theorems 13 and 12 give no indication of the existence of a tensor X with non-vanishing torsion that satisfies Eq. (15). However, when J and $\tilde{J}$ are SCK tensors, it turns out that even though $\tilde{J}^{-1}J$ solves Eq. (15), it is in general not torsionless.

Example 14. Suppose that (Q, g) is the Euclidean space with Cartesian coordinates (x, y) and let

$$J = \begin{bmatrix} x^2+1 & xy \\ xy & y^2 \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

J and $\tilde{J}$ are then SCK tensors, thus, according to Theorem 11, $X = \tilde{J}^{-1}J$ solves Eq. (15). But X has a non-vanishing torsion, since $(N_X)^1_{21} = 4x^2y \neq 0$.

4.4. *X diagonal*

We will now consider Eq. (15) under the assumption that X is diagonal in the coordinates (q^i). We seek a way to characterize all such solutions. Suppose that X is non-singular, $\det X \neq 0$, and diagonal in the coordinates (q^i), i.e.,

$$X^j_i = u^j \delta^j_i, \quad u^j = u^j(q^1, \ldots, q^n) \neq 0.$$

Moreover, Eq. (15) can be written as

$$0 = X\mathrm{d}Z^{(j)} + \mathrm{d}Z^{(j-1)} - Z^{(j)}\mathrm{d}Z^{(n-1)}, \quad j = 0, \ldots, n-2. \tag{19}$$

When written in components these equations become

$$0 = \sum_{1\le r_1<\cdots<r_{n-j}\le n} u^i \partial_i (u^{r_1}\cdots u^{r_{n-j}}) + \sum_{1\le r_1<\cdots<r_{n-j+1}\le n} \partial_i (u^{r_1}\cdots u^{r_{n-j+1}}) - \sum_{1\le r_1<\cdots<r_{n-j}\le n} u^{r_1}\cdots u^{r_{n-j}}\partial_i (u^1+\cdots+u^n), \quad j=0,\ldots,n-2, i=1,\ldots,n. \tag{20}$$

In the case when all diagonal elements of X are equal, i.e., $u^1 = \cdots = u^n =: u$, then X is a solution of (15) for arbitrary choice of the function u. In fact, all diagonal solutions can be constructed by taking smaller blocks of this kind, in the sense of Theorem 13.

Theorem 15. *Assume X is diagonal in the coordinates q^i, i.e., $X_i^j = u^j\delta_i^j$. Then X satisfies Eq. (15) if and only if (possibly after renumbering the eigenvalues u^j and the coordinates q^j simultaneously) X has the following block structure:*

$$X = \mathrm{diag}(X_1, X_2, \ldots, X_3), \quad X_k = \mathrm{diag}(\phi^k, \ldots, \phi^k) \tag{21}$$

where X_k is a quadratic matrix of size n_k and $\phi^k = \phi^k(q^{n_1+\cdots+n_{k-1}+1}, \ldots, q^{n_1+\cdots+n_k})$. In other words, X satisfies (15) if and only if the eigenvalues u^j are related in the following way:

$$\left.\begin{array}{r} \phi := u^{j_1} = \cdots = u^{j_r} \\ \phi \ne u^j, \quad j\ne j_k, \quad k=1,\ldots,r \end{array}\right\} \quad \Rightarrow \quad \phi = \phi(q^{j_1},\ldots,q^{j_r}),$$

where ϕ is an arbitrary function (sufficiently regular), depending only on the coordinates $q^{j_1},\ldots,q^{j_r}$.

Example 16 ($n = 3$). In order to clarify the complicated notation in Theorem 15, we specify in the case $n = 3$ all situations that may occur. Let $X = \mathrm{diag}(u^1, u^2, u^3)$. Eq. (15) is satisfied if and only if u^1, u^2, u^3 can be written in one of the following ways (up to permutations of the indices):

$$\begin{cases} u^1 = \phi(q^1,q^2,q^3) \\ u^2 = \phi(q^1,q^2,q^3) \\ u^3 = \phi(q^1,q^2,q^3)\end{cases} \quad \begin{cases} u^1 = \phi(q^1) \\ u^2 = \psi(q^2,q^3) \\ u^3 = \psi(q^2,q^3)\end{cases} \quad \begin{cases} u^1 = \phi(q^1) \\ u^2 = \psi(q^2) \\ u^3 = \omega(q^3)\end{cases}$$

where ϕ, ψ and ω are arbitrary functions.

Proof. We multiply the first of Eqs. (19) with X^{-1} and subtract it from the second equation. Thereafter, we multiply the modified second equation with X^{-1} and subtract it from third equation. When we proceed in this manner, we are left with the equations

$$0 = \mathrm{d}Z^{(m)} + \Big(-Z^{(m)}X^{-1} + Z^{(m-1)}X^{-2} + \cdots + (-1)^{m+1}Z^{(0)}X^{-(m+1)}\Big)\mathrm{d}Z^{(n-1)}, \quad m=0,\ldots,n-2 \tag{22}$$

By induction over m, it is a simple task to show that the i-th diagonal element of

$$Z^{(m)} - Z^{(m-1)}X^{-1} + \cdots + (-1)^m Z^{(0)}X^{-m}$$

consists of exactly those terms of $Z^{(m)}$ that depend on u^i. Thus, the i-th component of Eq. (22) can be written as

$$0 = \partial_i\left(\sum_{1\le r_1<\cdots<r_{n-m}\le n} u^{r_1}\cdots u^{r_{n-m}}\right) \tag{23}$$

$$-\left(\sum_{1\le r_1<\cdots<r_{n-m-1}\le n, r_s\ne i} u^{r_1}\cdots u^{r_{n-m-1}}\right)\partial_i(u^1+\cdots u^n). \tag{24}$$

After some manipulation, this equation becomes

$$0=\sum_{k=1,k\neq i}^{n}\left(\sum_{1\leq r_1<\cdots<r_{n-m-2}\leq n,r_s\neq i,k} u^{r_1}\cdots u^{r_{n-m-2}}\right)(u^i-u^k)\partial_i u^k.$$

For the sake of simplicity, we let $i=1$. The other cases follow by symmetry. If we collect all these equations for different m, we obtain the matrix equation

$$A[\partial_1 u^2\cdots\partial_1 u^n]^{\mathrm{T}}=0, \tag{25}$$

where A is a square matrix of size $n-1$, defined as

$$A_k^m=\sum_{1<r_1<\cdots<r_{n-m-2}\leq n,r_s\neq k} u^{r_1}\cdots u^{r_{n-m-2}}(u^i-u^k).$$

If we consider (25) as a linear system of equations for the functions $\partial_1 u^k$ and reduce it into triangular form, we obtain the equation $B[\partial_1 u^2\cdots\partial_1 u^n]^{\mathrm{T}}=0$, where

$$B=\begin{bmatrix} u^1-u^2 & u^1-u^3 & \cdots & u^1-u^n \\ 0 & (u^2-u^3)(u^1-u^3) & \cdots & (u^2-u^n)(u^1-u^n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (u^{n-1}-u^n)\cdots(u^1-u^n)\end{bmatrix}.$$

By renumbering the coordinates q^i and the functions u^i simultaneously, it is no restriction to assume that X has the form (21). The equation in row $n-n_t$ of (25) has the form

$$n_t(u^1-\phi^t)(u^2-\phi^t)\cdots(u^{n_1+n_2+\cdots+n_{t-1}}-\phi^t)\partial_1\phi^t=0.$$

Thus, $\partial_1\phi^t=0$, i.e., ϕ^t does not depend on q^1. In the same way we conclude that ϕ^t cannot depend on any of the variables $q^1,q^2,\ldots,q^{n_1+n_2+\cdots+n_{t-1}}$. If we change the order of the blocks in (21) by renumbering again the coordinates q^i and the functions u^i, we reach the desired result for the other blocks, which completes the proof. □

There exist solutions of Eq. (15) that do not belong to any of the families of solutions described in this section.

Example 17. Let (Q,g) be the Euclidean space with coordinates (x,y), and let

$$X=\frac{1}{x-y}\begin{bmatrix}1+x^2y & -1-y^3\\ 1+x^3 & -1-xy^2\end{bmatrix}.$$

The tensor X cannot be expressed on the form $\tilde{J}^{-1}J$ for SCK tensors J and $\tilde{J}$. This is easily realized by comparing coefficients of the polynomial elements in $(x-y)\tilde{J}X$ and $(x-y)J$, for arbitrary SCK tensors J and $\tilde{J}$. X has also a non-vanishing torsion and is not a direct sum of some simple solutions as in Theorem 13. Since this is valid also for X^{-1}, the tensor X cannot be reached by any of the methods for obtaining solutions, which has been described in this section.

5. Multiplication and Levi-Civita equivalence

It has been shown by Benenti in [1] that the concept of cofactor pair systems is closely connected to the theory of equivalent dynamical systems. In this section we will explain this connection and discuss the role of multiplication of cofactor pair systems in terms of equivalent dynamical systems.

Equivalence of dynamical systems was first introduced by Levi-Civita in [6]. We say that two dynamical systems (Q,g,F) and $(Q,\bar{g},\bar{F})$, on the same configuration space Q, are *equivalent* if they have the same trajectories but possibly different velocities. In other words, if solutions of one dynamical system are known, one can obtain solutions of an equivalent system by rescaling time in a proper way. In particular, two metrics g and $\bar{g}$ are equivalent if the dynamical systems $(Q,g,0)$ and $(Q,\bar{g},0)$, with no external forces, are equivalent. The equivalence of dynamical systems provides a method for solving a complicated dynamical systems by finding a simpler equivalent system which can be integrated by one of the known methods.

Solving dynamical systems is in general quite difficult, the most sophisticated methods are developed for Lagrangian systems, i.e., systems where the force F is conservative. Therefore, if we start with a non-Lagrangian dynamical system (Q, g, F), our chances of solving this system would certainly increase if we could find an equivalent Lagrangian system $(Q, \bar{g}, -\bar{\nabla} V)$. If we are able to find such a system, we could try to integrate it, for instance through separation of variables in the corresponding Hamiltonian–Jacobi equation. If we succeed with solving the Lagrangian system, then we are able to solve the original non-Lagrangian system by rescaling time.

Given a dynamical system, the problem of characterizing all equivalent systems is what we call the Levi-Civita equivalence problem. The general Levi-Civita equivalence problem is complicated and the existing characterizations hard to deal with. However, Benenti studied a stricter form of geodesic dynamical equivalence and was able to give a nice solution of the corresponding equivalence problem [1]. Two dynamical systems are *geodesically equivalent* if they are equivalent and their metrics also are equivalent. The most important of Benenti's results, in the context of this paper, is that a cofactor pair systems is in general equivalent to a separable Lagrangian system. In detail, if $\tilde{J} J^{-1}$ has real simple eigenvalues, then the cofactor pair system

$$\left(Q, g, F = -(\operatorname{cof} J)^{-1} \nabla V = -(\operatorname{cof} \tilde{J})^{-1} \nabla \tilde{V}\right)$$

is equivalent to a separable Lagrangian system

$$\left(Q, \bar{g}, \bar{F} = -\bar{\nabla} V = -(\operatorname{cof} \tilde{J} J^{-1})^{-1} \bar{\nabla} \tilde{V}\right),$$

where the metric $\bar{g}$ is defined through the SCK tensor J as $\bar{g}^{ij} = (\det J) J^{ij}$. We note that the metric $\bar{g}$ of the equivalent Lagrangian system may not be definite, even though the metric g of the cofactor pair system is positive definite.

Thus, given a pair of SCK tensors J and $\tilde{J}$ such that $\tilde{J} J^{-1}$ has real simple eigenvalues, we can use the multiplication described in Theorem 5 in order to produce large families of non-Lagrangian dynamical systems that are all geodesically equivalent to separable Lagrangian systems. For example, let us start with the trivial solution $V_\mu = \mu$ of Eq. (9). By taking powers,

$$V_{\mu,k} = \underbrace{\mu * \mu * \cdots * \mu}_{k \text{ times}},$$

with respect to the multiplication from Theorem 5, we obtain an infinite family of solutions of Eq. (9). For each k, the dynamical system

$$\left(Q, g, F = -(\operatorname{cof} J)^{-1} \nabla V_k^{(0)} = -(\operatorname{cof} \tilde{J})^{-1} \nabla V_k^{(n-1)}\right), \tag{26}$$

where $V_{\mu,k} = V_k^{(0)} + V_k^{(1)} \mu + \cdots + V_k^{(n-1)} \mu^{n-1}$, is a different cofactor pair system on the Riemannian manifold (Q, g). The dynamical system (26) is equivalent to the Lagrangian system

$$\left(Q, \bar{g}, \bar{F} = -\bar{\nabla} V_k^{(0)} = -(\operatorname{cof}(\tilde{J} J^{-1}))^{-1} \bar{\nabla} V_k^{(n-1)}\right),$$

which can be solved through separation of variables. By rescaling of time $t \to \bar{t}$, where $\mathrm{d}\bar{t}/\mathrm{d}t = \det J$, one can then obtain the general solution of the original system (26).

6. Examples

In this section we give some examples of QCR equations in order to illustrate how the multiplication works in some concrete cases.

Example 18 (*Jacobi Family of Separable Potentials, $n = 2$*). Let (Q, g) be the Euclidean space with the Cartesian coordinates $(q^1, q^2) = (x, y)$, and let the SCK tensors J and $\tilde{J}$ be given by:

$$J = \begin{bmatrix} -x^2 + \lambda_1 & -xy \\ -xy & -y^2 + \lambda_2 \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where λ_1 and λ_2 are constants. The fundamental equation is

$$\begin{aligned}0 &= D_J D_{\tilde{J}}\phi \\ &= xy\frac{\partial^2\phi}{\partial x^2} - xy\frac{\partial^2\phi}{\partial y^2} + (-x^2+\lambda_1+y^2-\lambda_2)\frac{\partial^2\phi}{\partial x\partial y} + 3y\frac{\partial\phi}{\partial x} - 3x\frac{\partial\phi}{\partial y}.\end{aligned}$$

We note that the fundamental equation in this example is a special case of the well known Bertrand–Darboux equation [12], that characterizes two-dimensional separable potentials. According to Theorem 3, $(V, \tilde{V})$ is a solution of the QCR equation if and only if $\tilde{V}$ satisfies the fundamental equation and V is the unique (up to an additive constant) function satisfying the equation $\nabla V = \mathrm{cof}\ J\nabla\tilde{V}$. When $(V, \tilde{V})$ and $(W, \tilde{W})$ solves the QCR equation, the multiplication generates a new solution according to the formula

$$(V, \tilde{V}) * (W, \tilde{W}) = (VW - (-\lambda_2 x^2 - \lambda_1 y^2 + \lambda_1\lambda_2)\tilde{V}\tilde{W},\ V\tilde{W} + \tilde{V}W - (-x^2 - y^2 + \lambda_1 + \lambda_2)\tilde{V}\tilde{W}).$$

By multiplying trivial solutions in the following way:

$$(V_k, \tilde{V}_k) := \Big((0,1)^2 + (\lambda_1\lambda_2, \lambda_1+\lambda_2)\Big) * (0,1)^{k-1}, \quad k = 1, 2, \ldots, \tag{27}$$

we obtain an infinite sequence of non-trivial solutions of the QCR equation. Note that $(V_k, \tilde{V}_k)$ are simply different powers of $(0, 1)$, but the constant terms in the first element are eliminated in order to obtain simpler expressions. Hence,

$$\Big(Q, g, F = -(\mathrm{cof}\ J)^{-1}\nabla V_k = -\nabla\tilde{V}_k\Big)$$

is an infinite sequence of Lagrangian systems with separable potentials. The functions $\tilde{V}_k$ are polynomials in x and y given by

$$\begin{aligned}\tilde{V}_1 &= x^2 + y^2 \\ \tilde{V}_2 &= (x^2+y^2)^2 - (\lambda_1 x^2 + \lambda_2 y^2) \\ \tilde{V}_3 &= (x^2+y^2)^3 - 2(x^2+y^2)(\lambda_1 x^2 + \lambda_2 y^2) + (\lambda_1^2 x^2 + \lambda_2^2 y^2) \\ &\ \ \vdots\end{aligned}$$

As an alternative to (27), we can define the functions $\tilde{V}_k$ recursively by

$$\tilde{V}_k = \Big[x^2 + y^2 - (\lambda_1+\lambda_2)\Big]\tilde{V}_{k-1} + \Big[\lambda_2 x^2 + \lambda_1 y^2 - \lambda_1\lambda_2\Big]\tilde{V}_{k-2},$$

for $k = 3, 4, \ldots$, where $\tilde{V}_1 = x^2 + y^2$ and $\tilde{V}_2 = (x^2+y^2)^2 - (\lambda_1 x^2 + \lambda_2 y^2)$. The sequence $\tilde{V}_k$ is nothing but the upper half of the bi-infinite Jacobi family of potentials (see [14]).

Proposition 19. *Let $\tilde{V}_0 := 1$. Then the functions $\{\tilde{V}_k\}$ form a basis for the space of polynomial solutions of the fundamental equation.*

Proof. The proof is technical and lengthy so we present here only the idea of the proof. Let ϕ be a polynomial in x and y, required to satisfy the fundamental equation. By inserting it into the FE, we obtain a polynomial equation in which we can identify coefficients at different powers $x^j y^k$ in order to obtain relations between the coefficients in ϕ. By analyzing these relations, it is possible (after lengthy calculations) to conclude that a general polynomial solution of the fundamental equation has the form

$$\phi = \sum_{a=0}^{n} A_{n-a}\sum_{m=a}^{n}\sum_{i=0}^{n-m}\binom{n-m}{i}\sum_{k=0}^{m-a}\binom{i+k-1}{k}\times\binom{n-a-1}{m-a-k}(\lambda_2-\lambda_1)^k(-\lambda_2)^{m-a-k}x^{2i}y^{2(n-m-i)},$$

where A_i are arbitrary constants. One can see that any basis for these polynomials contains no polynomials of odd order, and exactly one polynomial of each even degree. Since this holds also for the polynomial solutions $\tilde{V}_k$, these must form a basis for all polynomial solutions of the fundamental equation. □

In this example the multiplication has been applied directly to solutions of the QCR equations, since in the two-dimensional case it coincides with the μ dependent QCR equation. The next example will illustrate what happens in the case $n = 3$, where we must use the related μ dependent system in order to apply the multiplication formula.

Example 20 (*Parabolic Family of Separable Potentials, $n = 3$*). In this example we let J and $\tilde{J}$ be SCK tensors, in three-dimensional Euclidean space with coordinates $q = (x, y, z)$, given by

$$J = \begin{bmatrix} \lambda_1 & 0 & x \\ 0 & \lambda_2 & y \\ x & y & 2z \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are constants. We have $\det X_\mu = Z^{(0)} + Z^{(1)}\mu + Z^{(2)}\mu^2 + \mu^3$, where

$$\begin{aligned} Z^{(0)} &= -\lambda_2 x^2 - \lambda_1 y^2 + 2\lambda_1\lambda_2 z \\ Z^{(1)} &= \lambda_1\lambda_2 - x^2 - y^2 + 2(\lambda_1 + \lambda_2)z \\ Z^{(2)} &= \lambda_1 + \lambda_2 + 2z. \end{aligned}$$

Suppose that $V_\mu = V^{(0)} + V^{(1)}\mu + V^{(2)}\mu^2$ and $\tilde{V}_\mu = \tilde{V}^{(0)} + \tilde{V}^{(1)}\mu + \tilde{V}^{(2)}\mu^2$ are two solutions of Eq. (11). The multiplication formula for $V_\mu * \tilde{V}_\mu$ is obtained by reducing the polynomial $V_\mu \tilde{V}_\mu$ modulo $\det X_\mu$. First we collect terms in the product $V_\mu \tilde{V}_\mu$ that are of the same degree in μ,

$$\begin{aligned} V_\mu \tilde{V}_\mu = {} & V^{(0)}\tilde{V}^{(0)} + \left(V^{(0)}\tilde{V}^{(1)} + V^{(1)}\tilde{V}^{(0)}\right)\mu + \left(V^{(0)}\tilde{V}^{(2)} + V^{(1)}\tilde{V}^{(1)} + V^{(2)}\tilde{V}^{(0)}\right)\mu^2 \\ & + \left(V^{(1)}\tilde{V}^{(2)} + V^{(2)}\tilde{V}^{(1)}\right)\mu^3 + V^{(2)}\tilde{V}^{(2)}\mu^4. \end{aligned}$$

After that, we replace the monomials μ^3 and μ^4 with the congruent expressions modulo $\det X_\mu$ that are of degree less than three,

$$\begin{aligned} \mu^3 &\equiv -Z^{(0)} - Z^{(1)}\mu - Z^{(2)}\mu^2 \\ \mu^4 &\equiv Z^{(0)}Z^{(2)} + \left(Z^{(1)}Z^{(2)} - Z^{(0)}\right)\mu + \left((Z^{(2)})^2 - Z^{(1)}\right)\mu^2. \end{aligned}$$

We have then obtained $V_\mu * \tilde{V}_\mu$:

$$\begin{aligned} V_\mu * \tilde{V}_\mu = {} & V^{(0)}\tilde{V}^{(0)} - Z^{(0)}(V^{(1)}\tilde{V}^{(2)} + V^{(2)}\tilde{V}^{(1)}) + Z^{(0)}Z^{(2)}V^{(2)}\tilde{V}^{(2)} \\ & + \left[V^{(0)}\tilde{V}^{(1)} + V^{(1)}\tilde{V}^{(0)} - Z^{(1)}(V^{(1)}\tilde{V}^{(2)} + V^{(2)}\tilde{V}^{(1)}) + (Z^{(1)}Z^{(2)} - Z^{(0)})V^{(2)}\tilde{V}^{(2)}\right]\mu \\ & + \left[V^{(0)}\tilde{V}^{(2)} + V^{(1)}\tilde{V}^{(1)} + V^{(2)}\tilde{V}^{(0)} - Z^{(2)}(V^{(1)}\tilde{V}^{(2)} + V^{(2)}\tilde{V}^{(1)}) \right. \\ & \left. + ((Z^{(2)})^2 - Z^{(1)})V^{(2)}\tilde{V}^{(2)}\right]\mu^2. \end{aligned} \tag{28}$$

By starting with the trivial solution $V_{0,\mu} = \lambda_1\lambda_2 + (\lambda_1 + \lambda_2)\mu + \mu^2$, we obtain an infinite sequence, $V_{k,\mu} = V_k^{(0)} + V_k^{(1)}\mu + V_k^{(2)}\mu^2 := V_{0,\mu} * \mu^k$, of non-trivial solutions by multiplying with different powers of μ. Note that, when k is a negative integer, μ^k is defined by

$$\mu^k := (\mu^{-1})^{-k} \equiv -\left(\frac{Z^{(1)}}{Z^{(0)}} + \frac{Z^{(2)}}{Z^{(0)}}\mu + \frac{1}{Z^{(0)}}\mu^2\right)^{-k}.$$

Just as in the previous example, we obtain an infinite family of separable Lagrangian systems with potentials defined by $V_k := V_k^{(2)}$,

$$\begin{aligned} & \vdots \\ V_{-1} &= \frac{\lambda_1\lambda_2}{2z\lambda_1\lambda_2 - \lambda_2 x^2 - \lambda_1 y^2} \\ V_0 &= 1 \\ V_1 &= -2z \end{aligned}$$

$$V_2 = 4z^2 + (x^2 + y^2)$$
$$V_3 = -8z^3 - 4z(x^2 + y^2) - (\lambda_1 x^2 + \lambda_2 y^2)$$
$$\vdots$$

This sequence of separable potentials can also be found in [14] and is called the parabolic family.

7. Conclusions

Cofactor pair systems constitute a family of dynamical systems on Riemannian manifolds that possesses several nice properties. In particular, new cofactor pair systems can be generated from known ones with the use of the multiplicative structure on the solution space of the so-called quasi-Cauchy–Riemann equations $(\mathrm{cof} J)^{-1}\nabla V = (\mathrm{cof}\tilde{J})^{-1}\nabla\tilde{V}$.

In this work, we show that multiplication of the quasi-Cauchy–Riemann equation is a special case of a multiplication defined for a much larger family of systems of partial differential equations. Systems that allow this multiplication are characterized by the tensors $X_\mu = X + \mu I$ that solve the equation

$$X_\mu \nabla \det X_\mu = \det X_\mu \nabla \mathrm{tr}\, X_\mu. \tag{29}$$

We present large families of solutions of Eq. (29) and methods for generating new solutions from known ones. We also discuss the role of multiplication of cofactor pair systems for constructing Levi-Civita equivalent dynamical systems.

The content of this work indicates that the following questions may be worth examining in the future:

(1) We have already mentioned that systems more general than those treated in this paper admit a multiplication on the solution space. In fact, there exists a large class of systems of first-order linear PDE's, to be described, for which solutions can be built through the multiplication $*$.

(2) We have seen that the Cauchy–Riemann equations appear as a special case of a QCR equation and that the multiplication $*$ reconstructs multiplication of complex analytical functions. The question remains of whether some other elements from the rich theory of analytical functions have counterparts in the more general setting. In particular, since the multiplication makes it possible to form power series of simple solutions of the QCR equations, it would be interesting to characterize those solutions that can be represented as power series of some elementary solutions.

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